APPLICATIONS OF MODAL METHODS

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ESTIMATORS

For the bispectrum the estimator takes the general form

$$\mathcal{E} = \frac{\sum_{l_i m_i} G^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3} C^{-1}_{l_1 m_1 l_1' m_1'} C^{-1}_{l_2 m_2 l_2' m_2'} C^{-1}_{l_3 m_3 l_3' m_3'} \left(a_{l_1' m_1'} a_{l_2' m_2'} a_{l_3' m_3'} - 3 C_{l_1' m_1' l_2' m_2'} a_{l_3' m_3'} \right)}{\sum_{l_i m_i} G^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3} C^{-1}_{l_1 m_1 l_1' m_1'} C^{-1}_{l_2 m_2 l_2' m_2'} C^{-1}_{l_3 m_3 l_3' m_3'} G^{l_1' l_2' l_3'}_{m_1' m_2' m_3'} b_{l_1' l_2' l_3'}}$$

We can put this in a general form by defining

$$\langle \mathfrak{a}_{\wp} \rangle \; \equiv \; \langle a_{l_1 m_1} a_{l_2 m_2} ... a_{l_p m_p} \rangle$$

$$\mathfrak{C}^{-1}_{\wp \wp'} \equiv C^{-1}_{l_1 m_1, l_1' m_1'} ... C^{-1}_{l_p m_p, l_p' m_2' p}$$

Where $\wp$ represents the $\wp = \{l_1, m_1, l_2, m_2, ..., l_p, m_p\}$ degrees of freedom

ESTIMATORS

The estimator for a general polyspectrum is then defined as

$$\bar{\mathcal{E}} \equiv \frac{\sum_{\wp\,\wp'} \langle \mathfrak{a}_{\wp} \rangle \mathfrak{C}^{-1}_{\wp\wp'} \left(\mathfrak{a}_{\wp} - \mathfrak{a}^{lin}_{\wp} \right)}{\sum_{\wp\,\wp'} \langle \mathfrak{a}_{\wp} \rangle \mathfrak{C}^{-1}_{\wp\wp'} \langle \mathfrak{a}_{\wp} \rangle}$$

where $\mathfrak{a}^{lin}_{\wp}$ is the appropriate linear term

ESTIMATORS

We will now go one step further by defining the weighted vectors (and matrix)

$$\mathcal{A}_\wp = \frac{\langle \mathfrak{a}_\wp \rangle}{\sqrt{C_{l_1} C_{l_2} ... C_{l_p}}}, \qquad \mathcal{B}_\wp = \frac{\mathfrak{a}_\wp - \mathfrak{a}_\wp^{lin}}{\sqrt{C_{l_1} C_{l_2} ... C_{l_p}}}, \qquad \mathcal{C}_{\wp\wp'} = \frac{\mathfrak{C}_{\wp\wp'}}{\sqrt{C_{l_1} C_{l'_1} ... C_{l_p} C_{l'_p}}},$$

And we can then write the estimator in matrix form as

$$\bar{\mathcal{E}} = \frac{\mathcal{A}^T \mathcal{C}^{-1} \mathcal{B}}{\mathcal{A}^T \mathcal{C}^{-1} \mathcal{A}}$$

BASIS

If we then suppose the existence of an orthonormal basis

$$\sum_{\wp} \mathcal{R}_{n\wp} \mathcal{R}_{n'\wp} = \delta_{nn'} \quad (\mathcal{R}\mathcal{R}^T = I)$$

built from some separable functions $\mathcal{R} = \lambda \mathcal{Q}$

$$\mathcal{R} = \frac{\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3}}{v_{l_1} v_{l_2} v_{l_3}} R_{n l_1 l_2 l_3}$$

$$R_{n l_1 l_2 l_3} = \lambda_{nm} Q_{n l_1 l_2 l_3} (= q_i q_j q_k + 5 \text{ perms})$$

BASIS

Then we can decompose our theory representing it as a set of modal coefficients

$$\mathcal{A}_{\wp} = \sum_n \alpha_n R_{n\wp} \quad (\mathcal{A} = \mathcal{R}^T \alpha)$$

$$\alpha = \mathcal{R}\mathcal{A}$$

We will truncate our basis at some nmax so so we can also define a projection operator

$$\mathcal{P} = \mathcal{R}^T \mathcal{R}$$

And we take our theory to be completely described by this basis

$$\mathcal{P}\mathcal{A} = \mathcal{A}$$

SIMULATION

This method can be used to simulate maps with a given bispectrum and trispectrum

$$a_{lm} = a^G_{lm} + \frac{1}{6} F_{NL} a^B_{lm} + \frac{1}{24} G_{NL} a^T_{lm}$$

$$a^B_{lm} = \sum_{l_i m_i} \int Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} b_{l_1 l_2 l_3} \frac{a^G_{l_2 m_2}}{C_{l_2}} \frac{a^G_{l_3 m_3}}{C_{l_3}}$$

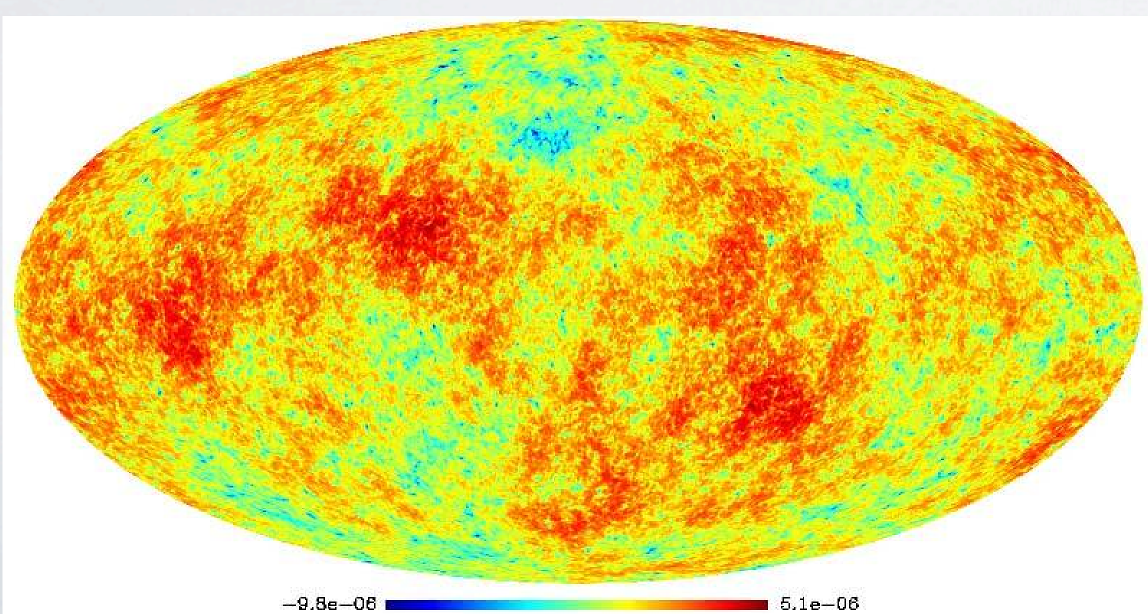
$$a^T_{lm} = \sum_{l_i m_i} \int Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} Y_{l_4 m_4} t_{l_1 l_2 l_3 l_4} \frac{a^G_{l_2 m_2}}{C_{l_2}} \frac{a^G_{l_3 m_3}}{C_{l_3}} \frac{a^G_{l_4 m_4}}{C_{l_4}}$$

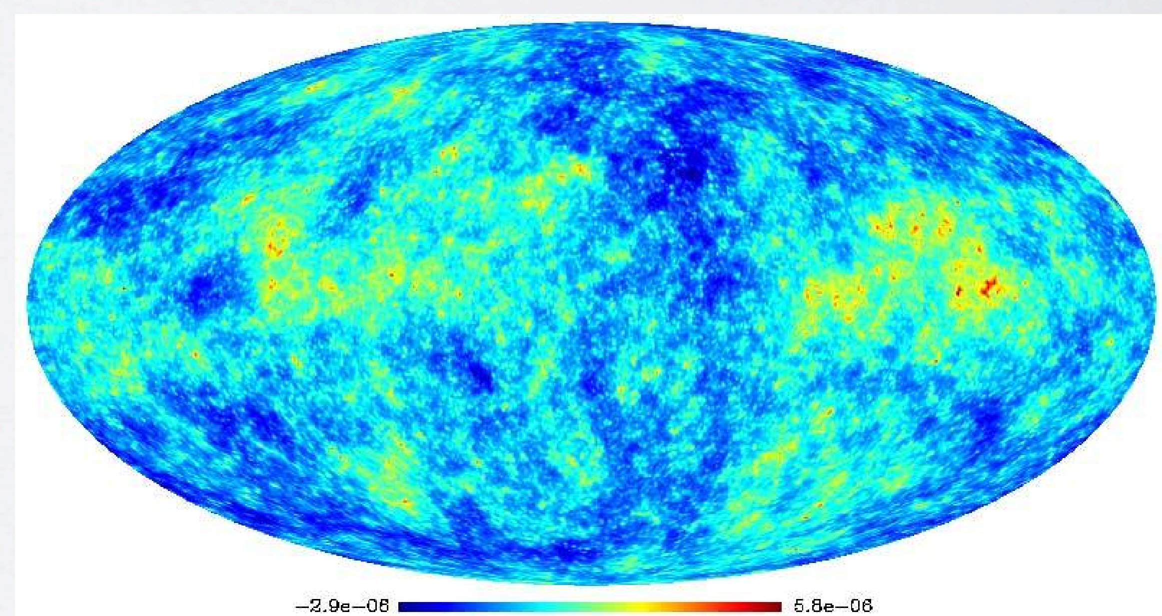
SIMULATION

Using the expansion the nonGaussian contributions can be easily calculated

$$a^B_{lm} = \sum_n \bar{\alpha}^Q_n \frac{q_l^{\{i}}{v_l\sqrt{C_l}} \int d^2\hat{\mathbf{n}} Y_{lm}(\hat{\mathbf{n}}) M^j(\hat{\mathbf{n}}) M^{k\}}(\hat{\mathbf{n}})$$

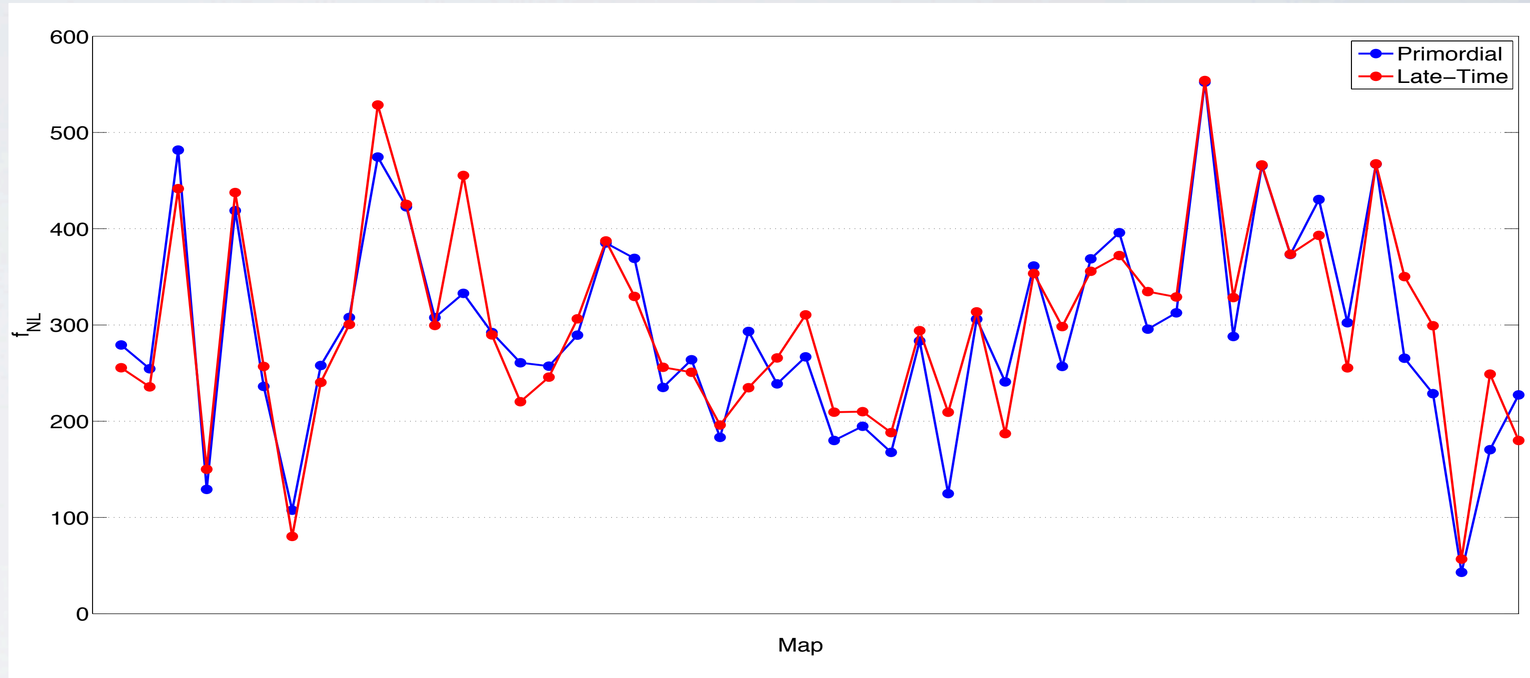
$$M^i(\hat{\mathbf{n}}) = \sum_{lm} \frac{q_l^i Y_{lm}(\hat{\mathbf{n}}) a^G_{lm}}{v_l\sqrt{C_l}}$$

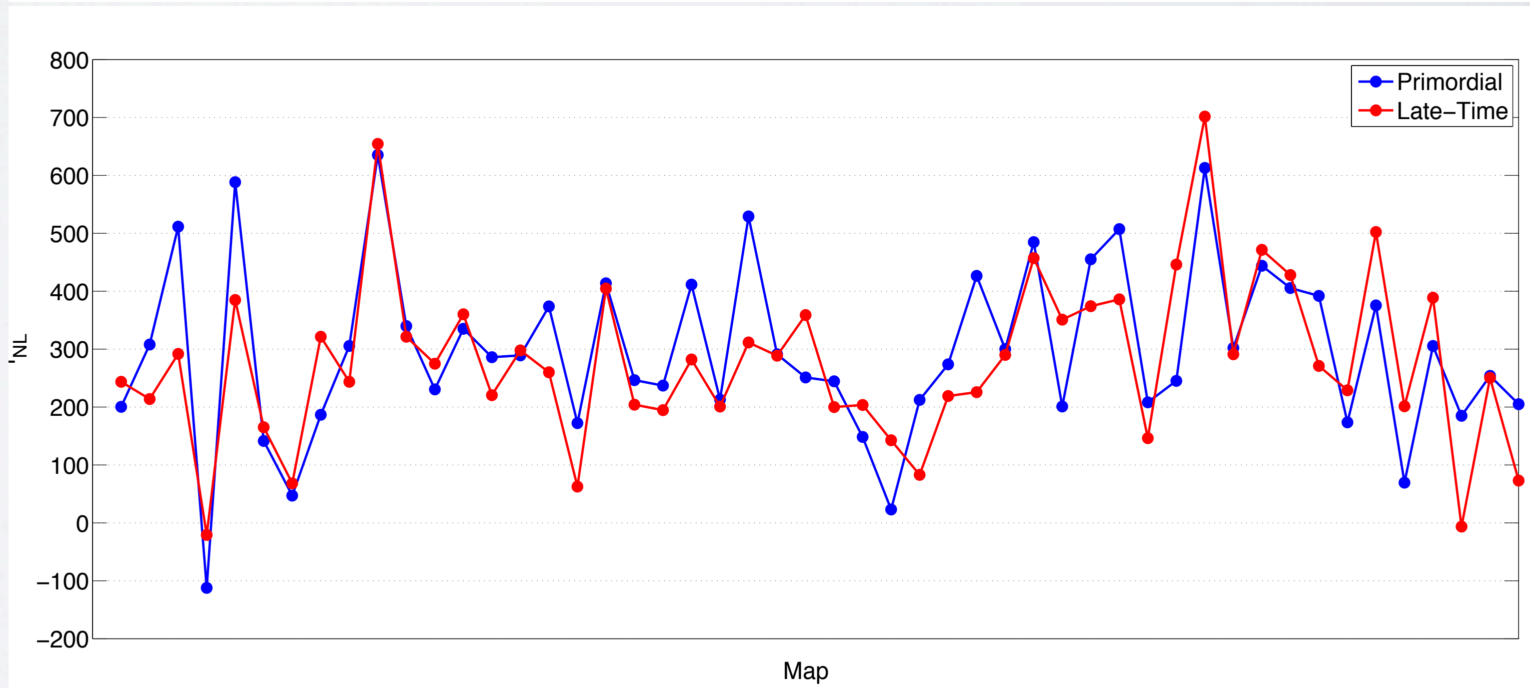




SIMULATION

To test the accuracy of the method we simulated maps using both the primordial and CMB decompositions and then applied both the primordial and CMB estimators to both sets to produce consistent results





	Ideal simulations		WMAP5 simulations	
	Average	St. Dev.	Average	St. Dev.
Primordial estimator	292.9	104.8	297.7	152.1
Late-time estimator	300.6	104.9	278.7	160
Internal st. dev.	38.5		102.6	

BASIS

We can perform the same modal decomposition on the data and the covariance

$$
\begin{aligned}
\alpha &= \mathcal{R}\mathcal{A} \\
\beta &= \mathcal{R}\mathcal{B} \longrightarrow \mathcal{P}\mathcal{B} = \mathcal{R}^T\beta \\
\zeta &= \mathcal{R}\mathcal{C}\mathcal{R}^T
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E} &\equiv \frac{\alpha^T\zeta^{-1}\beta}{\alpha^T\zeta^{-1}\alpha} \\
&= \frac{(\mathcal{R}\mathcal{A})^T\mathcal{R}\mathcal{C}^{-1}\mathcal{R}^T\mathcal{R}\mathcal{B}}{\mathcal{R}\mathcal{A}^T\mathcal{R}\mathcal{C}^{-1}\mathcal{R}^T\mathcal{R}\mathcal{A}} = \frac{\mathcal{A}^T\mathcal{P}\mathcal{C}^{-1}\mathcal{P}\mathcal{B}}{\mathcal{A}^T\mathcal{P}\mathcal{C}^{-1}\mathcal{P}\mathcal{A}}
\end{aligned}
$$

BASIS

We can understand the effect of the projection by considering

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{||} \\ 0 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} \mathcal{B}_{||} \\ \mathcal{B}_{\perp} \end{bmatrix} \qquad \mathcal{C}^{-1} = \begin{bmatrix} \mathcal{C}_{||}^{-1} & \mathcal{C}_{\times}^{-1} \\ {\mathcal{C}^{-1}}_{\times}^{T} & \mathcal{C}_{\perp}^{-1} \end{bmatrix}$$

$$\mathcal{X}_{||} \equiv \mathcal{P}\mathcal{X}$$

$$\mathcal{X}_{\perp} \equiv (I - \mathcal{P})\mathcal{X}$$

$$\mathcal{M}_{||} \equiv \mathcal{P}\mathcal{M}\mathcal{P}$$

$$\mathcal{M}_{\perp} \equiv (I - \mathcal{P})\mathcal{M}(I - \mathcal{P})$$

$$\mathcal{M}_{\times} \equiv \mathcal{P}\mathcal{M}(I - \mathcal{P})$$

BASIS

We can understand the effect of the projection by considering

$$\bar{\mathcal{E}} = \frac{\mathcal{A}_{||}\left(\mathcal{C}_{||}^{-1}\mathcal{B}_{||} + \mathcal{C}_{\times}^{-1}\mathcal{B}_{\perp}\right)}{\mathcal{A}_{||}^{T}\mathcal{C}_{||}^{-1}\mathcal{A}_{||}}$$

$$\mathcal{E} = \frac{\mathcal{A}_{||}\mathcal{C}_{||}^{-1}\mathcal{B}_{||}}{\mathcal{A}_{||}^{T}\mathcal{C}_{||}^{-1}\mathcal{A}_{||}}$$

The difference is the projection of contamination from the orthogonal space into the subspace

INVERSE COVARIANCE

Can we even calculate the covariance in the modal space?

Yes!

$$\zeta = \frac{1}{6}\left\langle \beta\beta^T \right\rangle$$

$$\begin{aligned}
\langle \beta_n \, \beta_{n'} \rangle = \sum_{l_i m_i l'_i m'_i} & \Bigg\langle \left(\frac{\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3}}{v_{l_1} v_{l_2} v_{l_3}} \frac{a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} - 3\, C_{l_1 m_1, l_2 m_2} a_{l_3 m_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3}}} R_{n l_1 l_2 l_3} \right) \\
& \times \left(\frac{\mathcal{G}^{l'_1 l'_2 l'_3}_{m'_1 m'_2 m'_3}}{v_{l'_1} v_{l'_2} v_{l'_3}} \frac{a_{l'_1 m'_1} a_{l'_2 m'_2} a_{l'_3 m'_3} - 3\, C_{l'_1 m'_1, l'_2 m'_2} a_{l'_3 m'_3}}{\sqrt{C_{l'_1} C_{l'_2} C_{l'_3}}} R_{n l'_1 l'_2 l'_3} \right) \Bigg\rangle \\
= \sum_{l_i m_i l'_i m'_i} & \frac{\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} \mathcal{G}^{l'_1 l'_2 l'_3}_{m'_1 m'_2 m'_3}}{v_{l_1} v_{l_2} v_{l_3} v_{l'_1} v_{l'_2} v_{l'_3}} R_{n l_1 l_2 l_3} \left[6 \langle a_{l_1 m_1} a_{l'_1 m'_1} \rangle \langle a_{l_2 m_2} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle \right. \\
+ \;& 9 \langle a_{l_1 m_1} a_{l_2 m_2} \rangle \langle a_{l'_1 m'_1} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle - 9\, C_{l_1 m_1, l_2 m_2} \langle a_{l'_1 m'_1} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle \\
- \;& \left. 9 \langle a_{l_1 m_1} a_{l_2 m_2} \rangle C_{l'_1 m'_1, l'_2 m'_2} \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle + 9\, C_{l_1 m_1, l_2 m_2} C_{l'_1 m'_1, l'_2 m'_2} \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle + \ldots \right] R_{n l'_1 l'_2 l'_3} \\
= 6 \sum_{l_i m_i l'_i m'_i} & \frac{\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} \mathcal{G}^{l'_1 l'_2 l'_3}_{m'_1 m'_2 m'_3}}{v_{l_1} v_{l_2} v_{l_3} v_{l'_1} v_{l'_2} v_{l'_3}} R_{n l_1 l_2 l_3} \frac{C_{l_1 m_1, l'_1 m'_1} C_{l_2 m_2, l'_2 m'_2} C_{l_3 m_3, l'_3 m'_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3} C_{l'_1} C_{l'_2} C_{l'_3}}} R_{n' l'_1 l'_2 l'_3}
\end{aligned}$$

INVERSE COVARIANCE

Also as all covariance matrices are symmetric positive definite they have a Cholesky decomposition

$$\zeta = \tilde{\lambda}\,\tilde{\lambda}^T$$

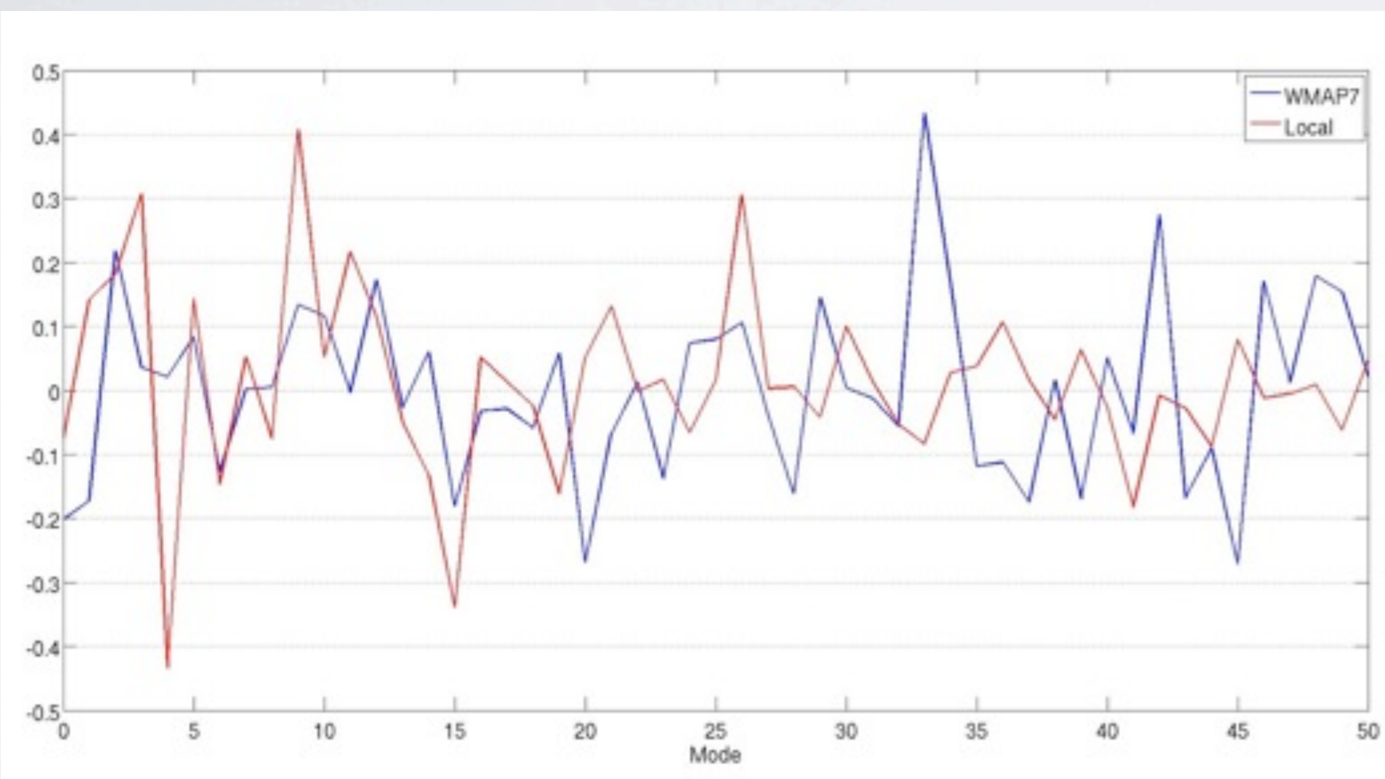
And we can absorb the covariance into our modes. This amounts to a re-orthogonalisation to an uncorrelated orthonormal basis

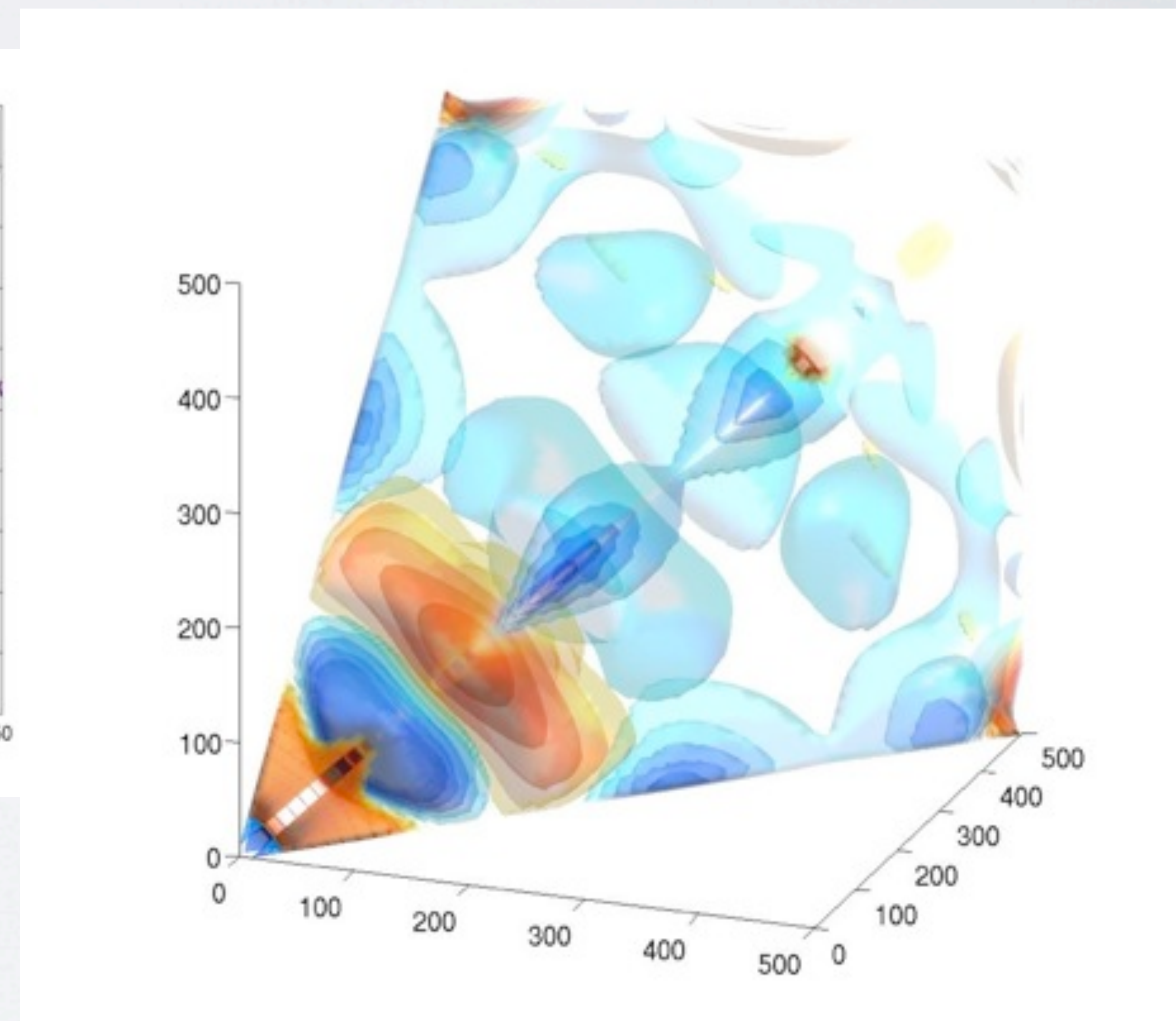
$$\alpha' = \tilde{\lambda}^{-1}\alpha \quad \beta' = \tilde{\lambda}^{-1}\beta$$

$$\mathcal{E} = \frac{\alpha'^T\beta'}{\alpha'^T\alpha'}\,, \quad \zeta' = I$$

RECONSTRUCTION

We have $\langle\beta\rangle = \alpha$ so can reconstruct the best fit bispectrum to the data by using the β as our α. If we have constructed a primordial basis as well then we can use the decomposition of projected primordial modes to find the best fit primordial bispectrum



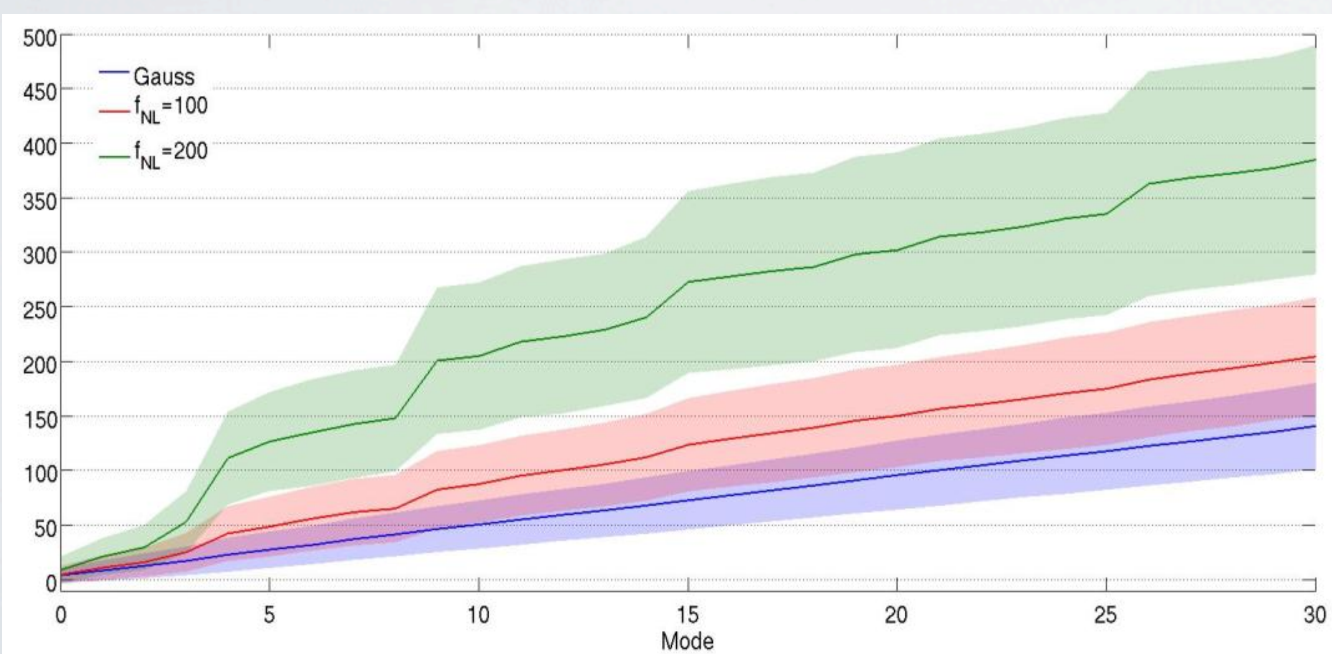


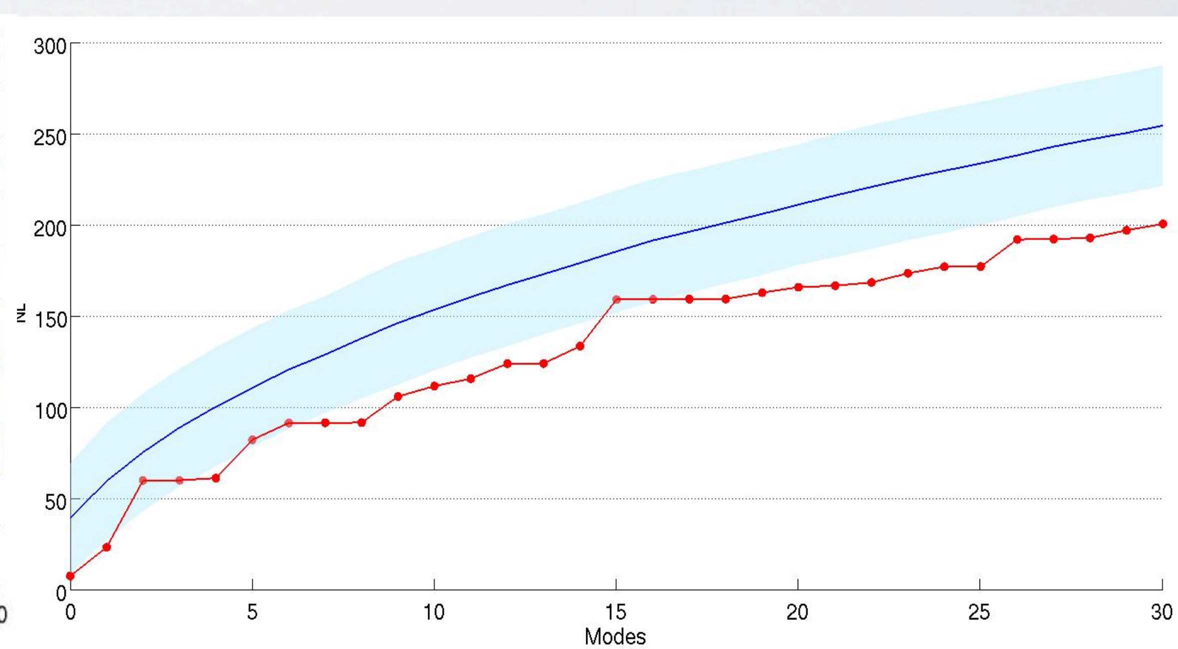
15

RECONSTRUCTION

In addition to constraining particular models we can perform a blind search

$$F_{NL}^2 = \frac{\beta'^T \beta'}{\alpha'^T \alpha'}$$





16

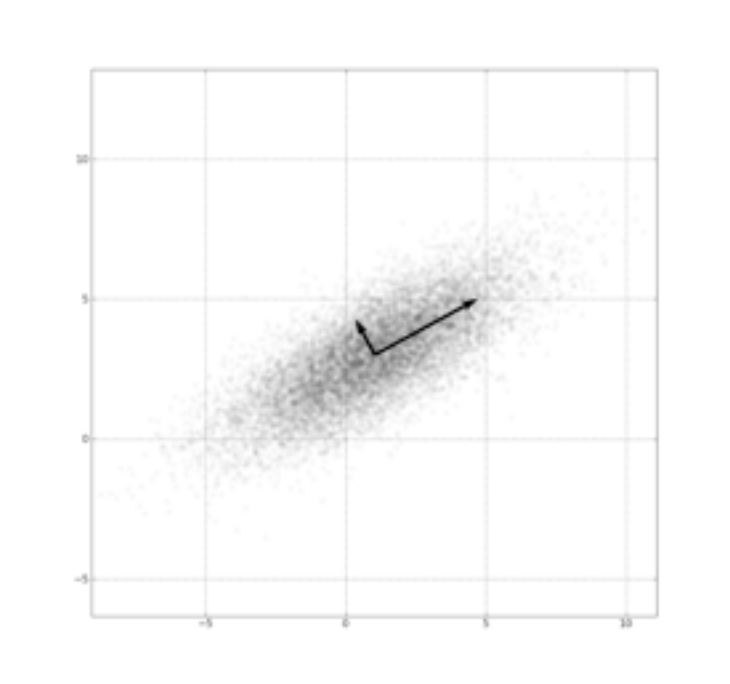
CONTAMINANTS

As we expect the covariance matrix to be the identity we can use principle component analysis to identify the shape of contaminants.

We first calculate the covariance matrix for beta from simulations

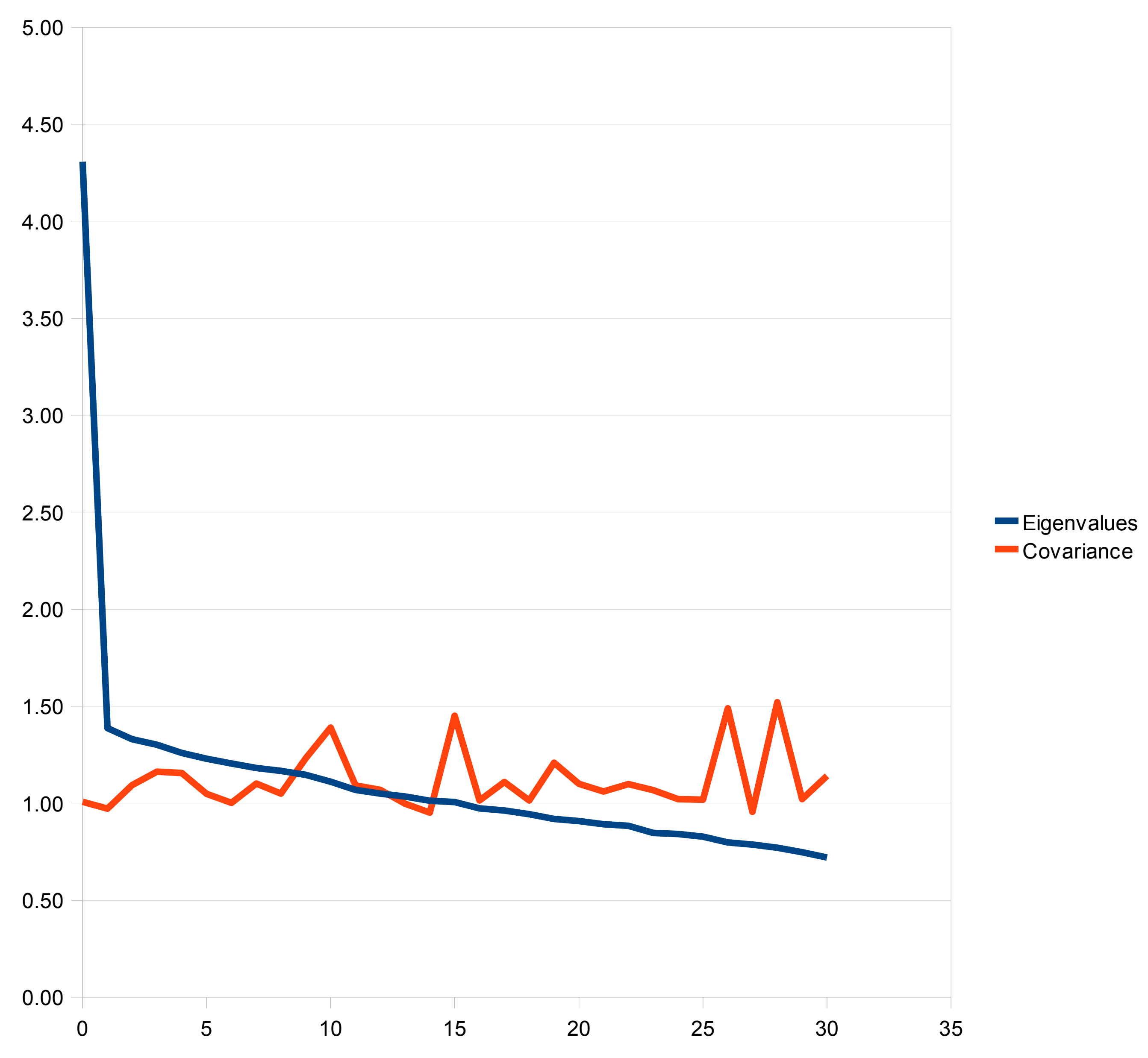
$$V \zeta V^T = D$$

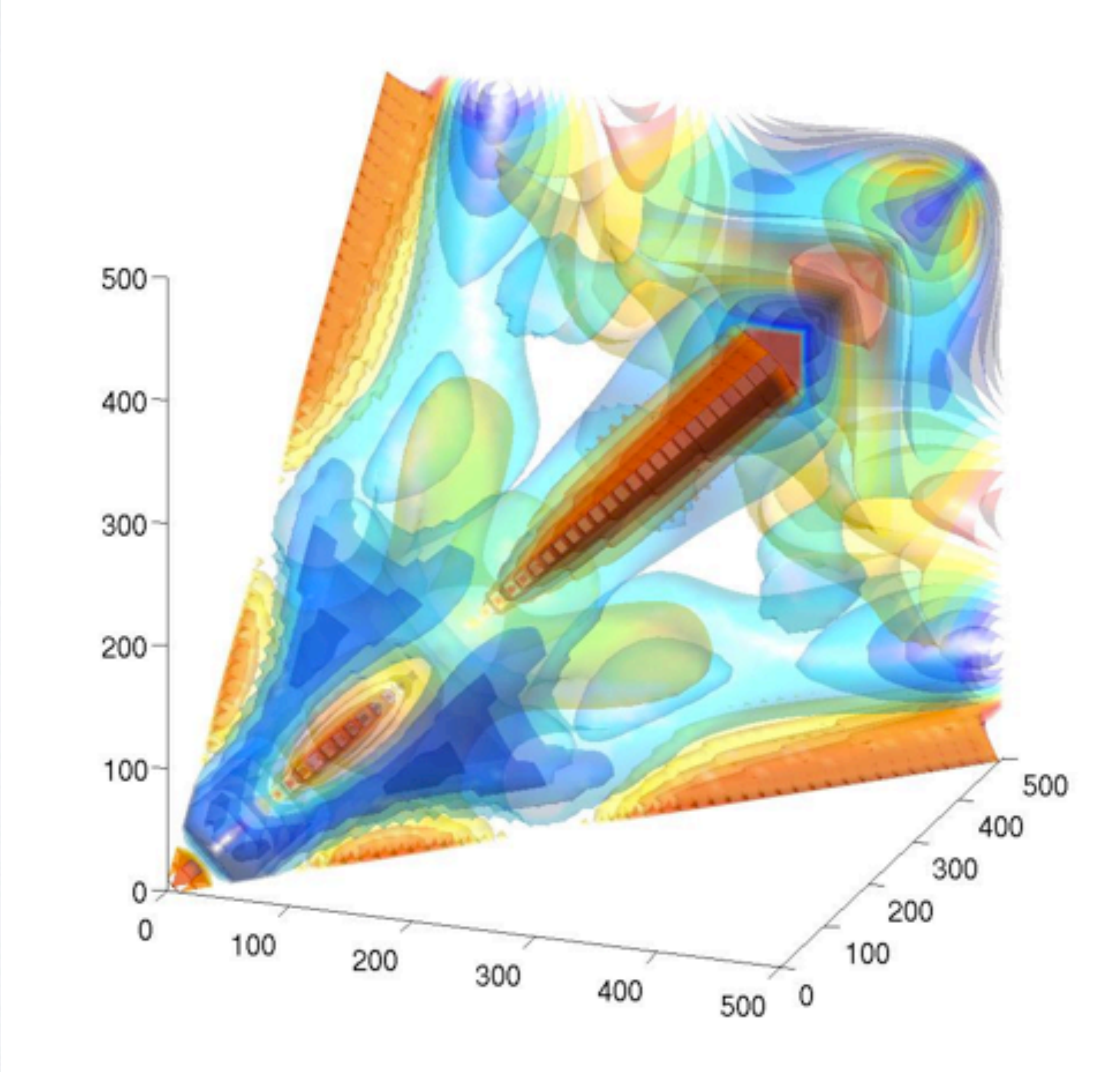
And then find the rotation which diagonalises it. This is equivalent to performing an eigen decomposition. The result is that you obtain a new orthonormal basis but now your modes are uncorrelated and ordered from greatest to least variance.



CONTAMINANTS

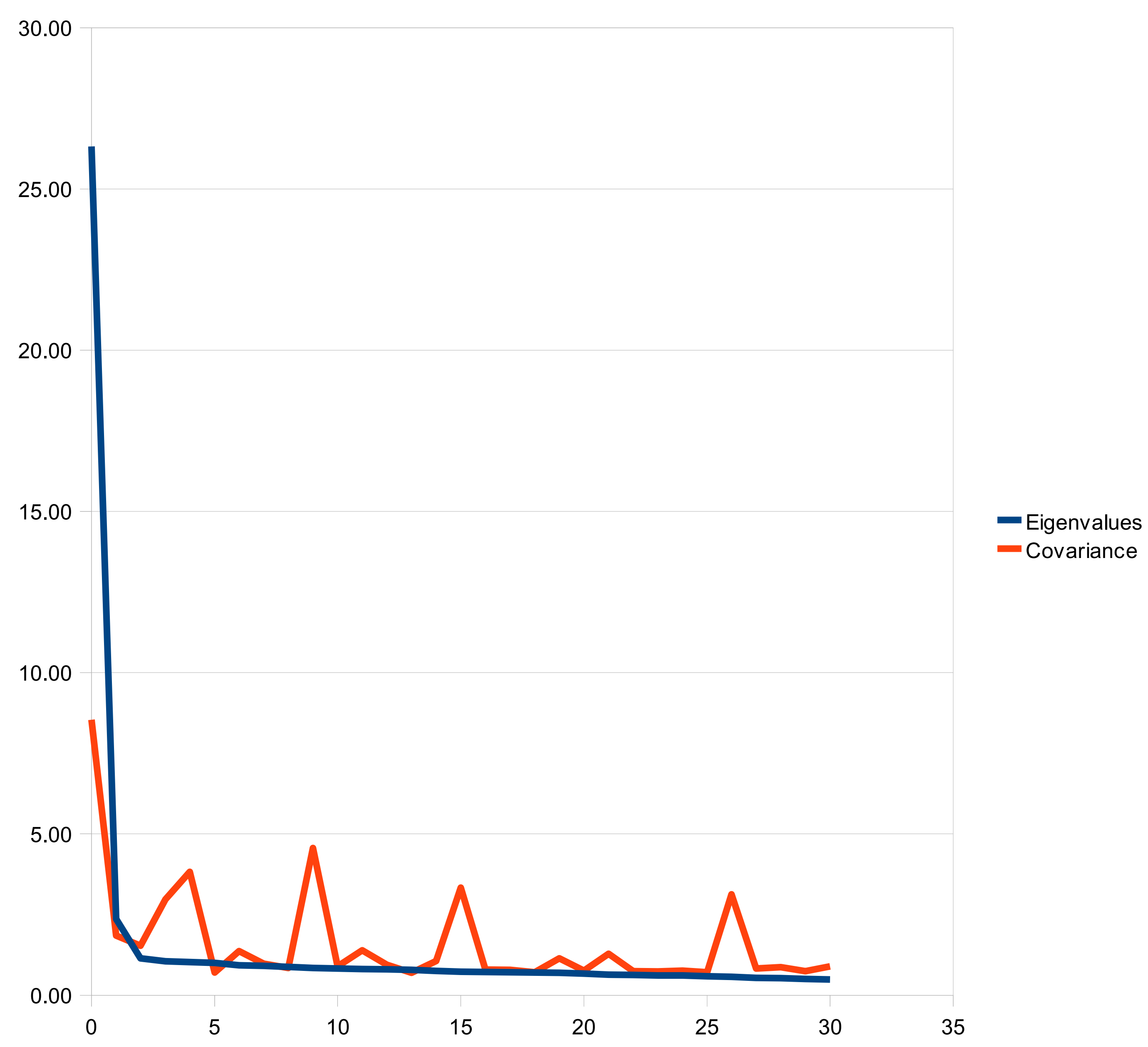
WMAP inhomogeneous noise

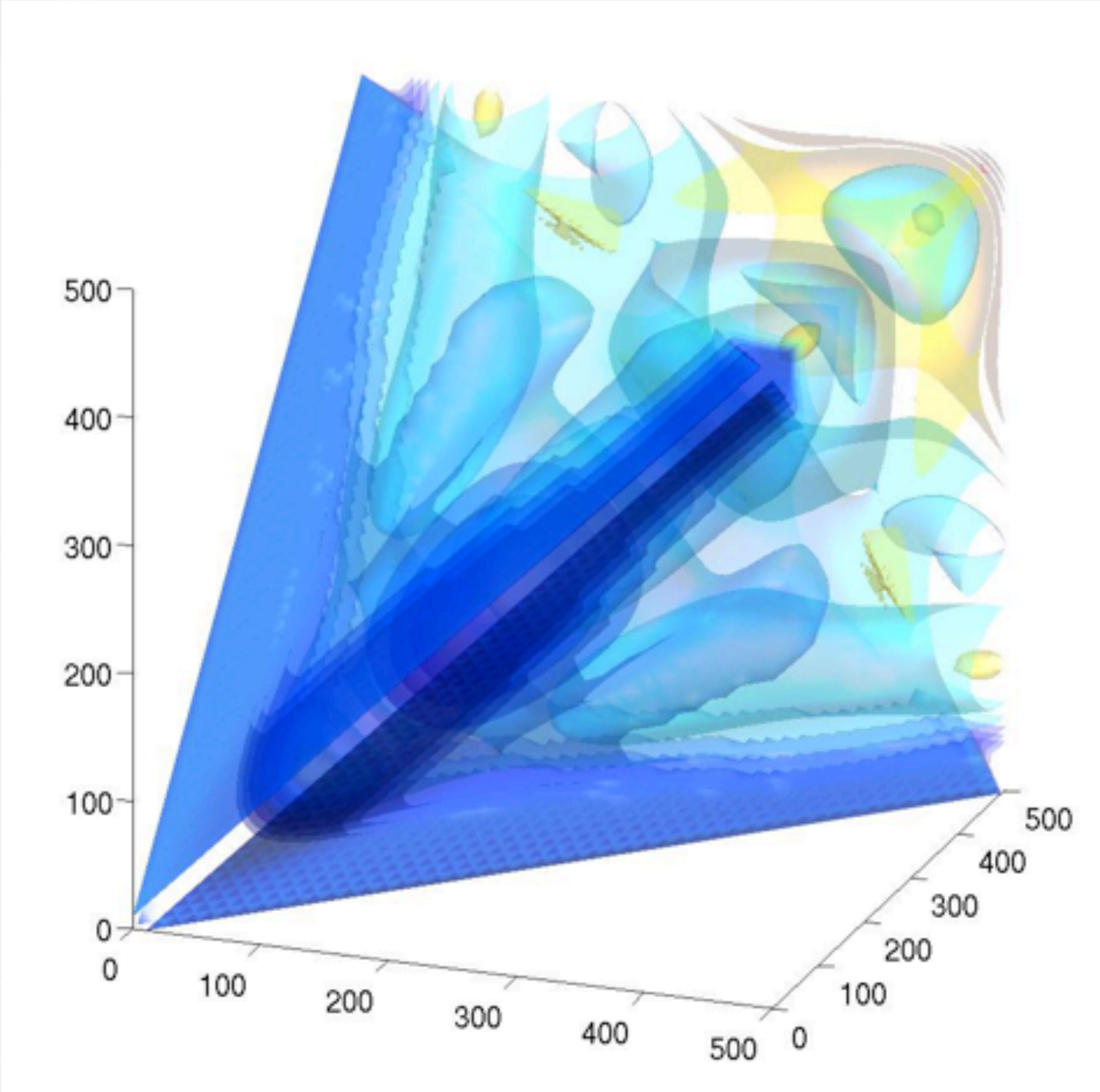




CONTAMINANTS

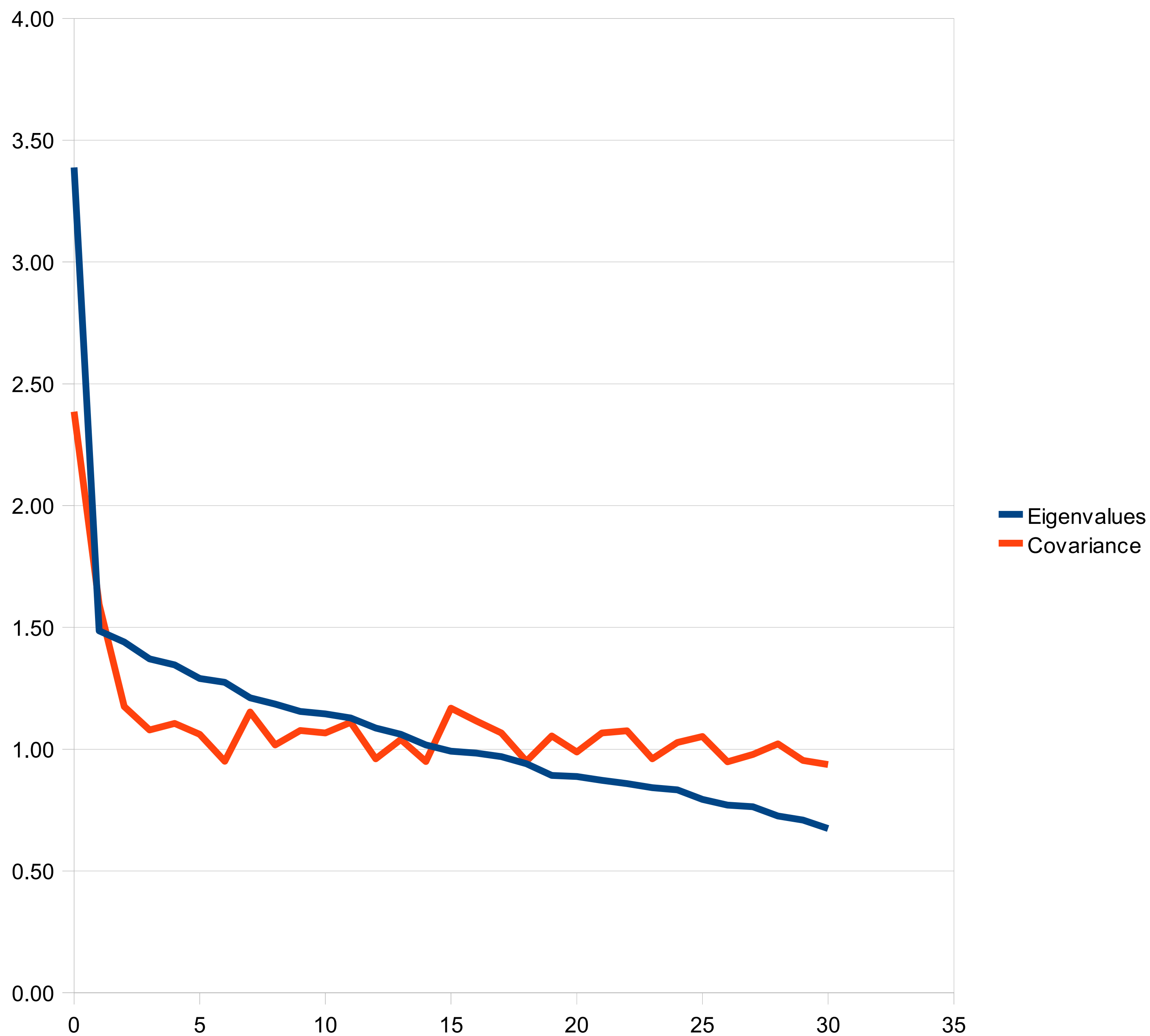
WMAP Mask

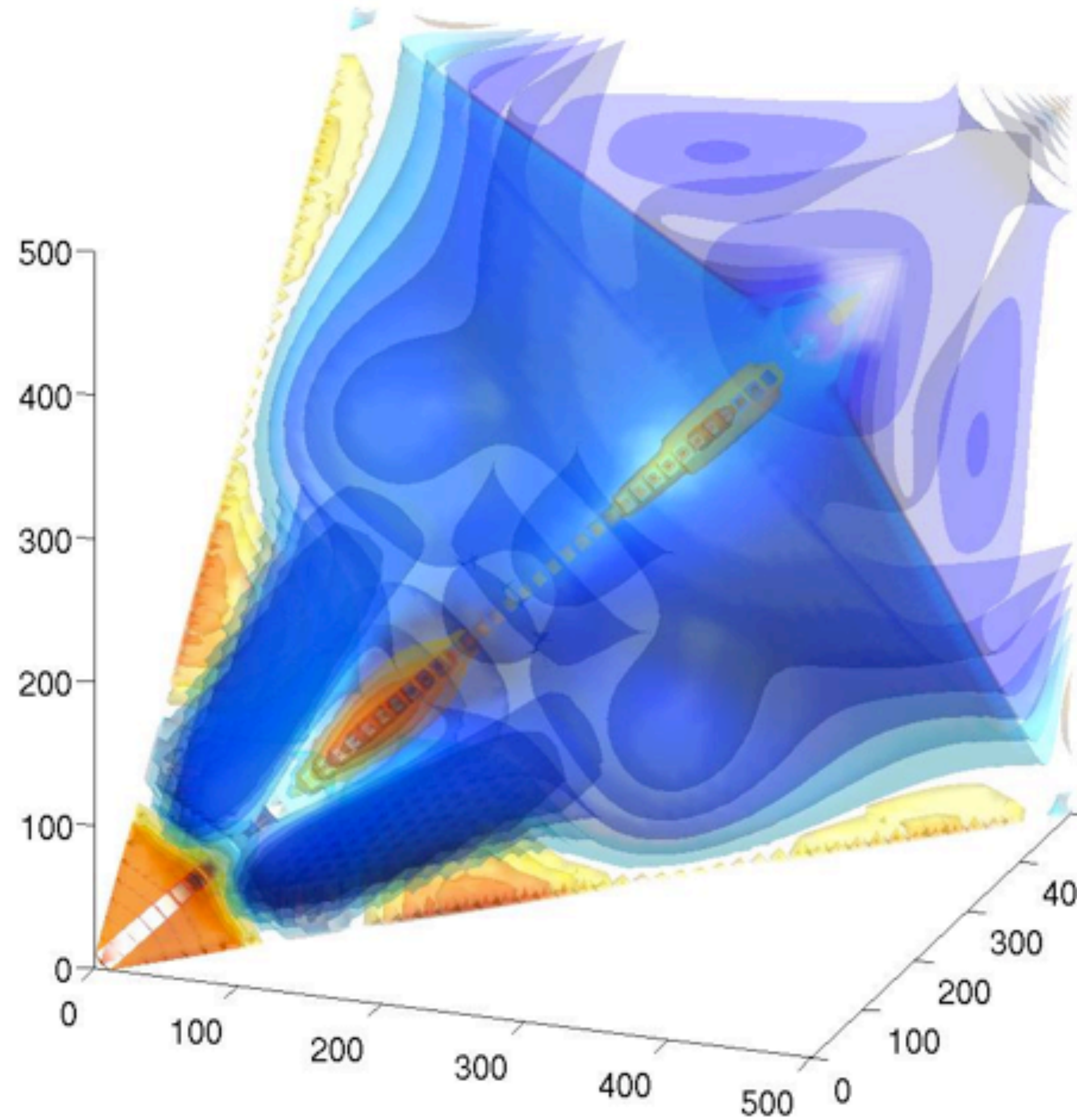




CONTAMINANTS

Point sources





CONCLUSIONS